3–15 Gauss's Law

Note: Gauss's Law is needed for AP Physics C: physical science majors, calculus based.

Flux: The amount of a field (gravity, electricity, or magnetism) passing through a defined area of space. $\phi = (Field\ \ Strength)(Area)$. Fields are visualized as lines (vectors) in space. The closer the lines are to each other the stronger the field in that region of space. Flux can then be thought of as the number of field lines passing through an area. The more lines passing through the area the greater the flux is. Flux is also greatest when the field lines pass through the area perpendicular to it. In Fig 15.1 no field lines pass through the area on the left. The field lines skim above and below it. In the area at an angle, in the middle, three of five field lines pass through, while in the identical area to the right five field lines pass through. The largest available area is exposed to the field when the area is perpendicular to the field. In this section we will deal with the electric flux. But, the same logic can be applied to gravitational flux and magnetic flux.

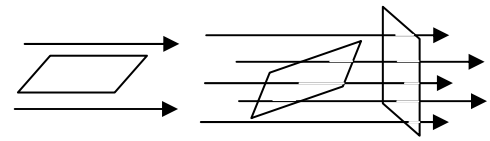

Fig 15.1

Electric Flux: Amount of electric field in an area of space. $\phi_E = \mathbf{E}\cdot\mathbf{A}$. This is a dot product of vectors, so $\phi_E = \mathbf{EA}\cos\theta$. But, this appears not to make any sense. Looking at the right area in Fig 15.1 we see that the area and electric field are 90° apart. Using the above formula would result in zero flux, in a configuration that was just described as having maximum flux. But, in physics ***<u>whenever you measure angles in relation to an area a normal is used</u>***. A normal is a line drawn perpendicular to the surface. This convention is used in force normal, it is used in flux, and it is used in optics for lenses and mirrors. Fig 15.2 shows the relationship between the area, the normal, and the electric field. Normal lines are drawn as dashed lines perpendicular to the surface. The angle between the normal and the field is 0° and this results in maximum flux.

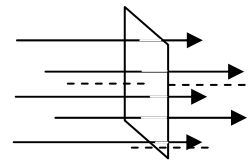

Fig 15.2

Flux Through a Cube (Closed Surface): The cube in Fig 15.3 has an electric field passing through it. There is zero flux through the top, bottom, front, and back of the cube, as the field does not pass through them. Through the left surface, the electric field is to the right and the normal is to the left. The angle between them is 180°. This results in a negative flux. ***<u>When ever a field enters an object the flux is negative</u>***. On the right side of the cube, the field is in the same direction as the normal. The angle between them is 0°. This results in a positive flux. ***<u>When ever a field leaves an object the flux is positive</u>***. The left area equals the right area, and the number of field lines passing through both areas is the same. The flux in (negative) cancels the flux out (positive). The **net flux** is zero.

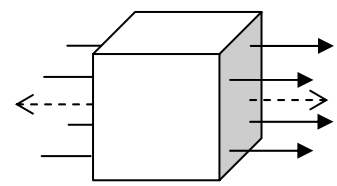

Fig 15.3

Irregular Closed Surfaces: If the area is irregular, then integral calculus must be used to find flux $\phi_E = \oint \mathbf{E}\cdot d\mathbf{A}$. The circle is added to the integral symbol to denote integration over a closed surface, such a sphere or the irregular shape in Fig 15.4. You are basically finding the flux over various small sections of area and summing them together.

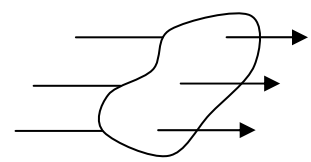

Fig 15.4

Gaussian Surface: Gauss was concerned with the net flux passing through a closed (**gaussian**) surface. Gaussian surfaces are any closed surfaces in space that are useful to analyze the problem. If you are looking at a point charge (spherical) then surround it with a spherical gaussian surface. When looking at a section of wire surround it with a cylindrical gaussian surface. Just like field lines it does not exist, it is a way to analyze invisible fields.

Analyzing a Proton Using a Gaussian Surface: In Fig 15.5 we analyze the electric field, at a distance r, surrounding a proton. ***<u>The diagram looks circular, but this is really a spherical arrangement.</u>*** Picture a gaussian sphere at a radius r surrounding the proton, use $\boxed{\phi_E = \oint \mathbf{E}\cdot d\mathbf{A}}$.

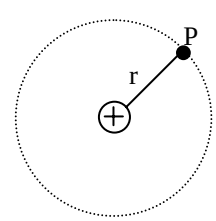

Fig 15.5

The symmetrical nature of this gaussian surface allows us to move E outside of the integral, $\phi_E = \mathbf{E}\oint d\mathbf{A}$. We know the surface area of a sphere, and can avoid its integration $\phi_E = \mathbf{E}(4\pi r^2)$.

Previously we solved for the electric field at a point point $\mathbf{E}=k\dfrac{q}{r^2}$, and we know that $k = \dfrac{1}{4\pi\,\epsilon_0}$.

Combining the three previous equation leads to $\phi_E = \left(\dfrac{1}{4\pi\,\epsilon_0}\dfrac{q}{r^2}\right)(4\pi r^2)$, which simplifies to $\boxed{\phi_E = \dfrac{q}{\epsilon_0}}$. Note that the flux is the same at any radius r. The flux equation gets larger by the square of r, but the electric field decreases by the square of r. When the equations combine r^2 cancels. The area grows but the proton's electric field is spread over a larger area. Thus the effect of increasing/decreasing r cancels. So $\phi_E = \oint \mathbf{E}\cdot d\mathbf{A}$ and $\phi_E = \dfrac{q}{\epsilon_0}$. Therefore, $\boxed{\phi_E = \oint \mathbf{E}\cdot d\mathbf{A} = \dfrac{q}{\epsilon_0}}$

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Gauss's Law: Gauss introduced the gaussian surface concept, and derived the previous formula $\boxed{\phi_E = \oint \mathbf{E}\cdot d\mathbf{A} = \frac{q}{\epsilon_0}}$. He also showed that this relationship works for any closed surface. It is easy to work with symmetrical objects that have known surface areas since you can avoid the actual integration. But, if the integration is performed over any irregular closed surface the relationship can be shown to hold true.

There are two ways to use a gaussian surface.

1. The surface can enclose at net charge. The proton in Fig. 15.5 was such an example. In these cases the above formula holds true. Stated formally: ***The net flux through any closed surface surrounding a point charge q is given by*** $\boxed{\phi_E = \frac{q}{\epsilon_0}}$. This can be a single proton or a single electron. It can also be a group of charges. You must work with the net charge enclosed by the surface. The logic is the same as that shown for the lone proton on the previous page.
2. The surface can enclose zero net charge. The charges could be outside the surface. In Fig. 15.6 the proton is outside the gaussian surface. Note that every field line entering the gaussian surface (negative flux) eventually exist the gaussian surface (positive flux) at another point. The positives and negative cancel. The net flux is zero. ***The net electric flux through a closed gaussian surface that surrounds no charge (or zero net charge) is zero***. Looking back at the cube in Fig. 15.3 we see that the net flux is zero through the cube (a closed gaussian surface). This holds true even if the surface is highly irregular.

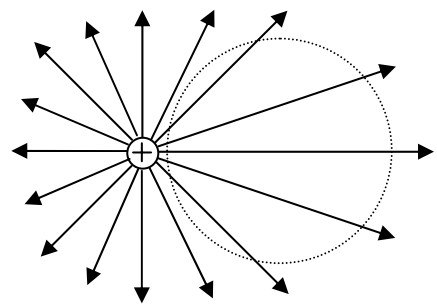

Fig 15.6

Example 15-1: Electric Flux Through Various Surfaces.

Fig 15.7 shows various gaussian surfaces drawn around several charges.

Find the electric flux through each surface.

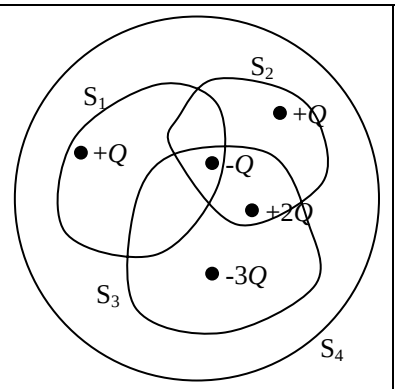

Fig 15.7

S_1: holds a $+Q$ and $-Q$ charge. The net charge is zero.

$$\phi_E = \frac{q}{\epsilon_0} \qquad \phi_E = \frac{(+Q)+(-Q)}{\epsilon_0} \qquad \boxed{\phi_E = 0}$$

S_2: holds a $+Q$, $-Q$, and $+2Q$ charge. The net charge is $+2Q$.

$$\phi_E = \frac{q}{\epsilon_0} \qquad \phi_E = \frac{(+Q)+(-Q)+(+2Q)}{\epsilon_0} \qquad \boxed{\phi_E = \frac{+2Q}{\epsilon_0}}$$

The positive flux indicates that the electric field through surface S_2 is moving outward, as it should from a positive charge.

S_3: holds a $+2Q$, $-Q$, and $-3Q$ charge. The net charge is $-2Q$.

$$\phi_E = \frac{q}{\epsilon_0} \qquad \phi_E = \frac{(+2Q)+(-Q)+(-3Q)}{\epsilon_0} \qquad \boxed{\phi_E = \frac{-2Q}{\epsilon_0}}$$

The negative flux indicates that the electric field through surface S_3 is moving inward, as it should from a negative charge.

S_4: holds a $+Q$, $+Q$, $+2Q$, $-Q$, and $-3Q$ charge. The net charge is zero.

$$\phi_E = \frac{q}{\epsilon_0} \qquad \phi_E = \frac{(+Q)(+Q)(+2Q)+(-Q)+(-3Q)}{\epsilon_0} \qquad \boxed{\phi_E = 0}$$

When viewed close up a collection of charges may seem complicated. Look at the three charges inside surface S_2. Close up we see $+Q$, $-Q$, and $+2Q$ charges. If we drew the electric field lines it would be a very complicated three-dimensional diagram. However, if we stepped way back, we would see a very small point with a +2Q net charge. At a great distance it would look like the electric field of a point charge. So when we draw a gaussian surface around these three charges we are analyzing them as a single point charge. The field lines that leave the system are those left over after tying the charges together.

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