1–11 Rotation Detailed, Rolling, and Angular Momentum

Note: This detail on rotation is needed for AP Physics C: physical science majors, calculus based.

Center of Mass: Objects rotate around a central axis and around a center of mass. It is therefore important to be able to locate the center of mass. The center of mass is each for shapes like squares, rectangles, circles, spheres, or equilateral triangles. It is in the middle. The following equation will find the center of mass of a system of point masses or for a system of geometric shapes (those just mentioned, that you can find the center of by inspection) $\boxed{\mathbf{r}_{cm} = \sum m\mathbf{r} / \sum m}$. Unusual shapes can be found experimentally by hanging the object from two or more positions, drawing vertical lines from the point of attachment of the string, and looking for an intersection. Or integral calculus can be used. For this course these last two methods will not be discussed.

Example 11-1: Center of Mass

Find the center of mass for the object in Fig 11.1a. It is a thin flat object composed of a rectangle (2m by 4m in length, mass 5 kg) and a square (2m long sides, mass 3 kg). Set up a coordinate axis system. For convenience place the coordinate axis at one corner of the object and divide the object into a rectangle and a square, as shown in Fig 11.1b. Find the center of mass the rectangle, relative to the coordinate axis, by inspection, $x = 1,\ y = 2$. Find the center of mass the rectangle, relative to the coordinate axis, by inspection, $x = 3,\ y = 1$. **Now you can pretend that the rectangle and square are point masses at these locations. The remainder of the problem is the method for solving for point masses in two dimensions.** You must work in each dimension separately.

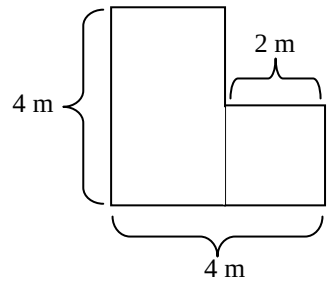

Fig 11.1a

$$x_{cm} = \frac{\sum mx}{\sum m} \qquad x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad x_{cm} = \frac{(5)(1)+(3)(3)}{(5)+(3)} = 1.75m$$

$$y_{cm} = \frac{\sum my}{\sum m} \qquad y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} \qquad x_{cm} = \frac{(5)(2)+(3)(1)}{(5)+(3)} = 1.63m$$

The center of mass is located at $\boxed{\mathbf{r}_{cm} = 1.75\mathbf{i}m + 1.63\mathbf{j}m}$

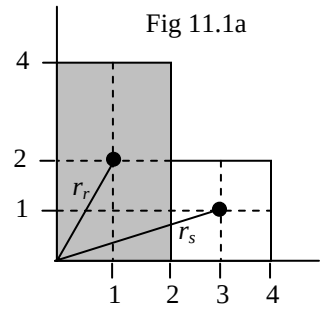

Fig 11.1b

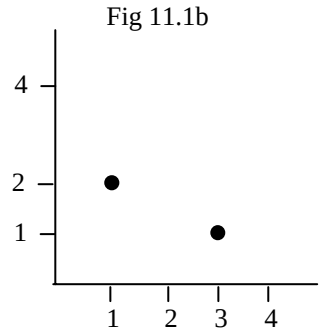

Fig 11.1c

Rotation: Since every point on a rotating object experiences a different tangential velocity displacement, velocity, and acceleration cannot be expressed in terms of meters. A particle on the outside edge of a rotating object covers a greater distance in the same time interval than a particle closer to the center. The only quantity that both points share in any given time interval is the angle through which they move, as shown to the right. In rotation we have to work in radians instead of degrees. This means that for every variable in linear (***translational***) motion there is a corresponding variable for rotation. And every equation in linear motion has a rotational counterpart. Displacement x is replaced by ***radians*** θ (radians). Velocity v is replaced by ***angular velocity*** ω (radians per second). Acceleration a is replaced by ***angular acceleration*** α (radians per second squared) The following three equations form a bridge between linear motion and rotation and should be memorized. $\boxed{x = r\theta}$ $\boxed{v = r\omega}$ $\boxed{a = r\alpha}$. The chart below, and on the following pages, compares rotation to linear motion. There is an analogous quantity and an analogous equation for rotation that parallels those learned in linear translational motion. Keep the three equations listed above in mind and become familiar with the new quantities.

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	Angular	Linear
Position	$\theta = \frac{arc\ length}{r}$	x
Displacement	$\Delta\theta = \theta - \theta_0$	$\Delta x = x - x_0$
Average Speed	$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$	$\bar{v} = \frac{\Delta x}{\Delta t}$ $\bar{v} = \frac{v_0 + v}{2}$
Instantaneous Speed	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$ $\omega = \frac{d\theta}{dt}$	$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ $v = \frac{dx}{dt}$ **Slope of displacement - time graph**
Average Acceleration	$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$	$\bar{a} = \frac{\Delta v}{\Delta t}$
Instantaneous Acceleration	**Tangential Acceleration** $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$ $\alpha = \frac{d\omega}{dt}$	$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ $a = \frac{dv}{dt}$ **Slope of velocity - time graph**
Kinematic Equations	$\boxed{\omega = \omega_0 + \alpha t}$ $\boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$ $\omega^2 = \omega_0^{\ 2} + 2\alpha(\theta - \theta_0)$	$\boxed{v = v_0 + at}$ $\boxed{x = x_0 + v_0 t + \frac{1}{2}at^2}$ $\boxed{v^2 = v_0^{\ 2} + 2a(x - x_0)}$
Tangential Speed	$\boxed{v = r\omega}$ $\omega = \frac{2\pi}{T}$	$v = \frac{2\pi r}{T}$
Centripetal Acceleration	**Radial Acceleration** $\boxed{a_c = \frac{v^2}{r} = \omega^2 r}$ Radial Acceleration is the acceleration directed along a radial (spoke) line. It is directed toward the center.	$a_c = \frac{v^2}{r}$
Inertia	**Moment of Inertia:** Depends on mass and distribution and thus varies for each object $\boxed{I = \int r^2 dm = \sum mr^2}$ Since these vary from object to object they are usually given. The three shown here are commonly used. The first one is the common shape for pulley, which are the most used. Cylinder: $\boxed{I = \frac{1}{2}MR^2}$ Cylindrical hoop: $I = MR^2$ Sphere: $I = \frac{2}{5}MR^2$	m
Force and Torque	**Torque:** Unbalance torques cause rotation. $\boxed{\tau = \mathbf{r} \times \mathbf{F}}$ $\boxed{\sum \tau = \tau_{net} = I\alpha}$	**Force:** Unbalanced forces cause translation. $\mathbf{F}$ $\boxed{\sum \mathbf{F} = \mathbf{F}_{net} = m\mathbf{a}}$
Kinetic Energy	$\boxed{K = \frac{1}{2}I\omega^2}$	$\boxed{K = \frac{1}{2}mv^2}$

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	Angular	Linear
Work	$W = \int \tau d\theta$ $W = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0{}^2$	$\boxed{W = \int \mathbf{F} \cdot d\mathbf{r}}$ **Area under force – distance curve** $W = \frac{1}{2} mv^2 - \frac{1}{2} mv_0{}^2$
Power	$\boxed{P = \frac{dW}{dt}}$ $\quad P = \tau\omega$	$\boxed{P = \frac{dW}{dt}}$ $\quad \boxed{P = Fv}$

Vector Product and Torque: Torque is a cross product of vectors. The **magnitude of a cross product is the area of the parallelogram formed by the contributing vectors**. The **direction of a cross product vector is determined by using the right hand rule**. So the direction of torque is out of the page for counterclockwise rotation, and into the page for clockwise rotations.

Translation vs. Rotation: Hit an object with a force directed into or out of the center of mass and it will translate (linear motion). Hit an object with a force perpendicular to a radial line extending from the center of mass and at the very edge of the object, and the object will rotate. Hit and object with a force between the center of mass and the edge and it will translate and rotate.

(Note: Planets and satellites follow circular motion, as they are not attached. Inner planets move faster as they are closer to the sun and must have larger tangential velocities. They also travel a shorter circumference. Thus they have shorter periods.)

Angular momentum: Masses that experience linear motion (translation) have velocity and thus have linear momentum. Rotating masses have angular velocity and thus have angular momentum. While linear momentum depends on mass and velocity, angular momentum depends on mass, mass distribution, and angular velocity. Think about it. In rotating objects the points of mass farther from the center are moving faster and thus have higher instantaneous momentum values than those closer to the center. Lots of mass, far from the center of mass, means higher angular momentum than the same mass, near the center of mass.

Angular momentum is conserved. The radius gets smaller, but angular velocity increases (vice versa as the skater moves arms outward). A galaxy, solar system, star, or planet forms from a larger cloud of dust. As the cloud is pulled together by gravity its radius shrinks. So the angular velocity must increase. These objects all begin to spin faster. That is why we have day and night.

	Angular	Linear
Momentum	$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p} = I\boldsymbol{\omega}}$	$\boxed{p = mv}$
Conservation of Momentum	$\mathbf{L}_i = \mathbf{L}_f$ $I\boldsymbol{\omega}_i = I\boldsymbol{\omega}_f$	$p_i = p_f$ $mv_i = mv_f$

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Example 11-2: Compound Bodies and Pulleys with Mass

A compound body consisting of, m_A = 6.0 kg, m_B = 8.0 kg, M_{pulley} = 1 kg, R_{pulley} = 0.10 m, is shown in Fig. 11.1a.

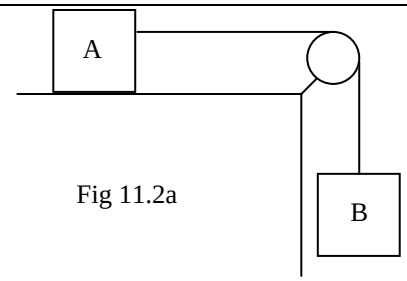

Fig 11.2a

What is the acceleration of the system?

There are three masses, so there are three FBD's, shown in Fig 11.2b. Make note of the interesting new FBD for a pulley. Gravity acts through the center and down, as usual. The normal force is created by the support pushing the pulley away from the table, and it follows the direction of the support through the center of the pulley. ***The tensions are tangent to the pulley. These tensions are a distance R (radius of pulley) from the center and they are perpendicular to the R. This provides the torque that rotates the pulley. Also note that there are two tensions. When we work with real pulleys that have mass the rope connecting the masses has different tensions in every separate segment.***

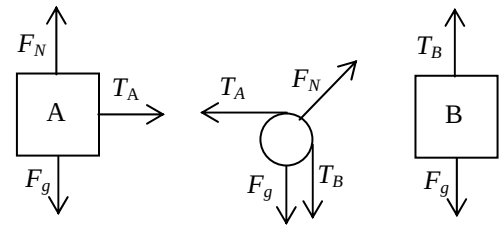

Fig 11.2b

Set up sum of force and sum of torque equations for relevant masses. As before use the direction of motion to assign positives and negatives. The direction of motion is to the right, clockwise, and then down.

$$\sum F_A = T_A \qquad \sum \tau = \tau_{cw} - \tau_{ccw} \qquad \sum F_B = F_g - T_B$$

$$m_A a = T_A \qquad I\alpha = R \cdot T_B - R \cdot T_A \qquad m_B a = m_B g - T_B$$

$$T_A = m_A a \qquad I\frac{a}{R} = R \cdot T_B - R \cdot T_A \qquad T_B = m_B g - m_B a$$

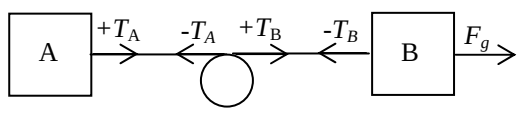

Fig 11.2c

Combine the three equations above to get

$I\frac{a}{R} = R\left(m_B g - m_B a\right) - R\left(m_A a\right)$ Substitute in the moment of inertia of a cylindrical disk (pulley) $I = \frac{1}{2}MR^2$

$\left(\frac{1}{2}MR^2\right)\frac{a}{R} = R\left(m_B g - m_B a\right) - R\left(m_A a\right)$ Cancel out the pulleys radius, group all expression with a, and simplify.

$$\frac{1}{2}Ma = m_B g - m_B a - m_A a \qquad m_A a + m_B a + \frac{1}{2}Ma = m_B g \qquad \boxed{a = \frac{m_B g}{\left(m_A + m_B + \frac{1}{2}M\right)}}$$

This looks familiar. If we did the problem the old way, with a massless pulley, we would look at it as linear, like Fig 11.2c

It would be $\sum F_{total} = F_{gB}$ $\qquad \left(m_A + m_B\right)a = m_B g \qquad a = \frac{m_B g}{\left(m_A + m_B\right)}$

This is identical except for the expression for half the pulley's mass. Can we just do all pulley problems the old way and just add a $\frac{1}{2}M$ to all the regular masses in the denominator. It seems to work, but you might loose points for not showing work.

And the $\frac{1}{2}M$ only works with pulleys that have a moment of inertia of $I = \frac{1}{2}MR^2$. If it were a spherical pulley, would we add $\frac{2}{5}M$ to the denominator, since its moment of inertia is $I = \frac{2}{5}MR^2$. Verify it on your own and see.

Fig 11.2c shows the problem sketched linear. Any forces that are perpendicular to the direction of motion were removed from this sketch. Vectors pointing in the direction of motion are noted with positive signs and those opposing motion are negative. It is apparent that tension cancels as before. However, unlike previous work in forces the pulley is not erased as it now has mass. It must be accounted for. Is there a shortcut method using this linear sketch that would show adequate supporting work?

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Example 11-3: Rolling Down an Incline

A spherical mass rolls 2 m down an incline shown in Fig 11.3a. The FBD for the sphere is shown in Fig 11.3b. In figure 11.3c the gravity and normal force vectors have been summed and the component of force down the slope is shown. $F_g\sin\theta$ pulls the sphere down the incline in the usual manner. However, the friction force vector is a distance R from the center of the sphere. These two vectors are perpendicular. This creates an unbalanced torque on the sphere, which causes it to rotate. The combined motion of rotating and moving down the slope is rolling.

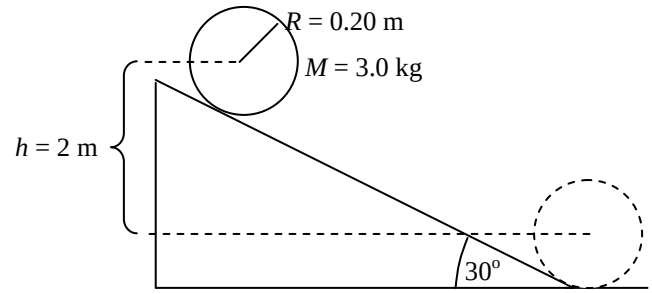

Fig 11.3a

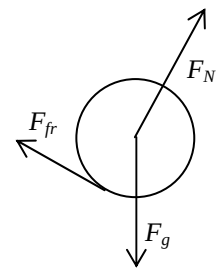

Fig 11.3b

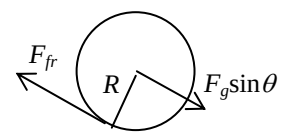

Fig 11.3c

What is the kinetic energy of the sphere at the bottom of the incline? The sphere is translating and rotating at the same time. The total kinetic energy is the addition of the translational and rotational kinetic energies.

$$K_{total} = K_{translation} + K_{rotation}$$

$$K_{total} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

It is also equal to the potential energy that converted in kinetic energy. Given the quantities in the problem, this is easiest to solve

$$K_{total} = mgh \qquad K_{total} = (3.0)(9.8)(2.0) = \boxed{58.8J}.$$

How fast is the sphere going at the bottom of the incline?

Now the first kinetic energy equation has relevance, $K_{total} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. Combine this with the moment of inertia equation of a sphere $\frac{2}{5}MR^2$, and the equation $\omega = \frac{v}{R}$ which converts angular values into linear values.

$$K_{total} = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 \qquad K_{total} = \frac{1}{2}Mv^2 + \frac{2}{10}Mv^2 \qquad K_{total} = \frac{7}{10}Mv^2$$

Rearrange for velocity, plug in values, and solve.

$$v = \sqrt{\frac{10}{7}\frac{K_{total}}{M}} \qquad v = \sqrt{\frac{10}{7}\left(\frac{58.8}{3.0}\right)} = \boxed{5.29\,m/s}$$

Another expression can be derive here also $v = \sqrt{\frac{10}{7}gh}$

What is the linear acceleration of the sphere down the inline?

The length down the ramp is $hyp = \frac{h}{\sin\theta}$

$$v^2 = v_0^{\,2} + 2a(x - x_0)$$

$$\left(\sqrt{\frac{10}{7}\frac{K_{total}}{M}}\right)^2 = (0)^2 + 2a((x) - 0)$$

$$\left(\sqrt{\frac{10}{7}\frac{Mgh}{M}}\right)^2 = (0)^2 + 2a\left(\frac{h}{\sin\theta}\right)$$

$$a = \frac{5}{7}g\sin\theta \qquad a = \frac{5}{7}(9.8)\sin 30^o = \boxed{3.5\,m/s^2}$$

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